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Phase-Shift Steer a Uniform Circular Array in the Fourier-Series Domain

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***Abstract*—Taylor's 1952 Fourier series for a uniform circular array's pattern here yields simple steering, with minimal distortion, by applying a phase ramp to the weights' DFT coefficients rather than to the weights themselves as for a linear array. Computational examples include steering a simple monopulse difference beam created as the difference between steered beams.**

I. Introduction

In 1952 Taylor wrote the complex embedded element pattern of a uniform circular array (UCA) as a Fourier series in azimuth, which formulation made the Fourier-series coefficients [1] for the UCA pattern itself just the element-pattern coefficients times the periodically extended discrete Fourier transform (DFT) of the array's complex combining weights. Modern optimization renders obsolete his resulting simple approach to pattern synthesis, but the Fourier formulation itself offers insights into array parameter selection and, the focus here, yields a method of translating a predesigned UCA pattern in azimuth by an arbitrary amount. Computation requires little more than taking a small discrete Fourier transform (DFT), and the approach yields only minor sidelobe errors when the array's number of design degrees of freedom—the number of element weights—is well matched to the array structure.

After next presenting the approach's simple mathematics, in Section III we provide computational examples using realistic element patterns obtained through electromagnetic simulation.

II. The Mathematics

A. Taylor's Fourier-Series Pattern Formulation

When polarization directions are defined as usual relative to the direction of propagation, the complex co-pol (or cross-pol) pattern for an M element UCA can be written as

$$f(\theta) = \sum_{m=0}^{M-1} w_m\, f_m(\theta) \tag{1}$$

where pattern elevations are implicit but fixed and identical and where UCA elements are indexed on $\{0, \ldots, M-1\}$.

In a UCA, any embedded element pattern $f_m(\theta)$ is a translation of any other. Taking element 0 as a prototype then,

$$f_m(\theta) = f_0(\theta - 2\pi m/M). \tag{2}$$

Substituting a doubly infinite Fourier series

$$f_0(\theta) = \sum_k a_k\, \mathrm{e}^{jk\theta} \tag{3}$$

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for the prototype pattern into (2), using the result in (1), reordering, and defining W_k for convenience yields

$$f(\theta) = \sum_k W_k\, a_k\, \mathrm{e}^{jk\theta} \tag{4}$$

$$W_k \triangleq \sum_{m=0}^{M-1} w_m\, \mathrm{e}^{-j2\pi km/M}\,. \tag{5}$$

The Fourier-series coefficients $W_k a_k$ of the UCA pattern are the coefficients a_k of the prototype element pattern scaled by the array weights' DFT coefficients W_k, which are periodic in k with period M. This is unsurprising, as (1) and (2) show array pattern $f(\theta)$ to be the convolution of the periodic discrete-angle array-weight sequence w_m with the continuous-angle prototype element pattern $f_0(\theta)$. This makes (4) just an ordinary Fourier convolution property for this specialized case mixing a discrete domain and a continuous, periodic one. The mixture manifests in spectral representation (4) as spectrum factor W_k being periodic in k while spectrum factor a_k is not.

Taylor presented this derivation in [1] and then proposed synthesizing desired product $W_k a_k$, for M consecutive values of k, by simply dividing out known coefficients a_k to obtain sequence W_k. Weights were then trivially obtained with an inverse DFT. He used two facts. First, $|a_k|$ rolls off to zero for large k because the embedded element pattern is bounded in magnitude. Second, fixing the M coefficients W_k effectively fixes the UCA pattern when well-chosen array geometry makes the rolloff fast enough. We'll use the same facts to steer.

B. A Method of Approximately Steering the Pattern

By (4) the UCA pattern steered by angle θ_s takes form

$$f(\theta - \theta_s) = \sum_k W_k P_k\, a_k\, \mathrm{e}^{jk\theta} \tag{6}$$

with steering factor $P_k = \mathrm{e}^{-jk\theta_s}$ for all $-\infty \le k \le \infty$. However, for product $W_k P_k$ to represent legitimate steered weights, to be their DFT, it must be periodic in k with period M. Ideal steering factor $\mathrm{e}^{-jk\theta_s}$ has that periodicity only when steering angle θ_s is an integral multiple of $2\pi/M$, which corresponds to the trivial case of steering by circularly shifting the weights.

1) Periodicity Using Coset Decomposition: Suppose we wish to construct a sequence with period 4. Partitioning the integer sample indices into the four subsets on these four lines,

$$\begin{array}{ccccccc} \ldots & -4 & & 0 & & 4 & \ldots \\ \ldots & & -3 & & 1 & & 5\ \ldots \\ \ldots & & & -2 & & 2 & \ldots \\ \ldots\,-5 & & & & -1 & 3 & \ldots \end{array}$$

shows that periodicity gives the sequence the same value at all sample indices in a line, so constructing the sequence amounts to choosing its value at one point per line. This breaking into lines amounts to coset decomposition, a standard topic in elementary group theory[2]. We next formalize it just enough.

Steering factor P_k must be invariant to changes in k by multiples of period M, by elements of the subset $M\mathbb{Z}$ of the integers $\mathbb{Z}$. So let us partition $\mathbb{Z}$ into this family of M cosets—translations—of that subset:

$$\mathbb{Z}/M\mathbb{Z} \triangleq \{M\mathbb{Z}, M\mathbb{Z}+1, \ldots, M\mathbb{Z}+(M-1)\}.$$

Group-theory notation $\mathbb{Z}/M\mathbb{Z}$ denotes such a "quotient group" or "factor group." To give P_k period M, we give it one value for all k in each coset, including "zero coset" $M\mathbb{Z}$. We choose a "coset representative" k from each of the M cosets, gather the M coset representatives into a set $[\mathbb{Z}/M\mathbb{Z}]$, also standard notation, fix P_k for $k \in [\mathbb{Z}/M\mathbb{Z}]$ to fix M values of the product $W_k P_k$ in (6), and let periodicity take care of the others.

But what coset representatives should we choose?

2) Steer the Largest Fourier Coefficient: To steer the product $W_k P_k$ that likely matters most, set $P_k = \mathrm{e}^{-jk\theta_s}$ in (6) for the k in each coset for which Fourier coefficient $W_k P_k a_k$ in (6) has the largest magnitude $|W_k a_k|$. (Sound array geometry will often make a_k negligible on k in the rest of the coset.) Again, this fully defines P_k through its required periodicity.

To this end, let us assign to $[\mathbb{Z}/M\mathbb{Z}]$ the k from each coset for which $|W_k a_k|$ is the largest. Most often $|W_k a_k|$ rolls off more or less monotonically with $|k|$ once $|k|$ is well away from zero, so this procedure will typically fill $[\mathbb{Z}/M\mathbb{Z}]$ with the smallest-magnitude k values, and the M largest values for $|W_k a_k|$ will be for these k. Our periodic steering factor is now

$$P_k = \mathrm{e}^{-jk'\theta_s} \tag{7}$$

$$\text{with } k' \in [\mathbb{Z}/M\mathbb{Z}] \tag{8}$$

$$\text{and } k-k' \text{ a multiple of } M. \tag{9}$$

Such a k' exists, as k must be in some coset and so separated by multiples of M from the coset's other elements including its coset representative, which we can call k'. There is only one such k', because if there were two, by (9) they would differ by a multiple of M and so be in the same coset, while by (8) they would both be in $[\mathbb{Z}/M\mathbb{Z}]$, a contradiction as by construction each coset has only one coset representative. Steering factor P_k is periodic with period M because (7), (8), and (9) imply $P_{k+M} = \mathrm{e}^{-jk'\theta_s}$ using the same k', as $k-k'$ being a multiple of M in (9) implies $k+M-k'$ is also a multiple of M.

3) Computing the new steered weights: Replacing ideal but unrealizable steering factor P_k in (6) with our new but periodic steering factor P_k transforms (6) into approximate pattern

$$f'_{\theta_s}(\theta) \triangleq \sum_k \left(W_k\, \mathrm{e}^{-jk'\theta_s}\right) a_k\, \mathrm{e}^{jk\theta} \tag{10}$$

where the dependence of k' on k is not shown but is most certainly still present. Comparison to (4) and (5) lets us write

$$W_k\, \mathrm{e}^{-jk'\theta_s} = \sum_{m=0}^{M-1} \langle \text{steered weight} \rangle_k\, \mathrm{e}^{-j2\pi km/M}$$

from which an inverse DFT yields

$$\langle \text{steered weight} \rangle_k = \frac{1}{M} \sum_{k=0}^{M-1} \left(W_k\, \mathrm{e}^{-jk'\theta_s}\right) \mathrm{e}^{j2\pi km/M}$$

The tricky part is computing k' from k. Here $0 \le k < M$, so by (9) this inverts the map from k' to k given by $k = k' \bmod M$.

Is this map invertible? By construction $[\mathbb{Z}/M\mathbb{Z}]$ contains one element from each coset, so no two elements of $[\mathbb{Z}/M\mathbb{Z}]$ can differ by a multiple of M, as that would put them in the same coset. It follows that they cannot have the same value modulo M, so the map is one-to-one. It is also a map onto $\{0, \ldots, M-1\}$, as each element of the latter is in some coset, which in turn has a coset representative in $[\mathbb{Z}/M\mathbb{Z}]$. So map $k' \bmod M = k$ is invertible. In practice we can simply create an inversion lookup table of M entries. For each $k' \in [\mathbb{Z}/M\mathbb{Z}]$ we can compute $k = k' \bmod M$ and set up table entry $L(k) = k'$.

The lookup table assumes $0 \le k < M$, but its periodic extension to all k as needed in (10) is trivial: $L(k) \triangleq L(k \bmod M)$.

4) A Simple Error Bound: Ideal steered pattern (6) and realizable pattern (10) have identical sum terms for $k \in [\mathbb{Z}/M\mathbb{Z}]$, so the pattern difference and its obvious bound are

$$f(\theta-\theta_s) - f'_{\theta_s}(\theta) = \sum_{k \notin [\mathbb{Z}/M\mathbb{Z}]} W_k \left(\mathrm{e}^{-jk\theta_s} - \mathrm{e}^{jk'\theta_s}\right) a_k\, \mathrm{e}^{jk\theta}$$

$$\left| f(\theta-\theta_s) - f'_{\theta_s}(\theta) \right| \le 2 \sum_{k \notin [\mathbb{Z}/M\mathbb{Z}]} \left| W_k a_k \right|.$$

The integers in $[\mathbb{Z}/M\mathbb{Z}]$ are typically contiguous with the k just above and below $[\mathbb{Z}/M\mathbb{Z}]$ dominating the bound.

III. Simulation

A. Array Geometry and Element Design

We illustrate UCA steering and key design issues with two UCAs of the Fig. 1 geometry. For each we obtained the prototype embedded element pattern using simulation in CST Microwave Studio with one element driven and the others terminated[3]. The two Fig. 2 element-pattern magnitudes show a slightly broader element pattern in the $M = 49$ case, which is natural given the closer element spacing.

B. Optimal UCA Patterns for Computational Experiments

For each UCA we optimized the unsteered element pattern $f(\theta)$ of (1) using a second-order cone program (SOCP) to minimize receive noise while upper bounding sidelobe levels [4] at -15 dB, a bound barely active in Fig. 2. Performance in noise is characterized by taper losses for the $M = 31$ and $M = 49$ designs respectively of 0.08 dB and 0.27 dB relative to the theoretical best across any choice of M weights given the particular element design. Going from $M = 31$ to $M = 49$ improved the pattern only a little. We'll see why later. The designs are quite ordinary, but they will illustrate key ideas.

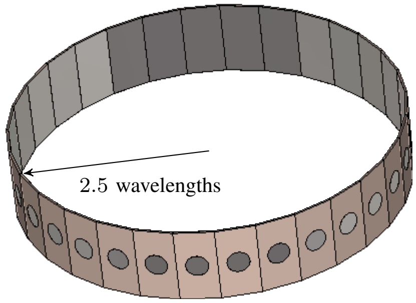


Fig. 1. The two UCA designs feature $M=31$ and $M=49$ vertically polarized patch elements. Circumferential spacing of 2.5 wavelengths $\times 2\pi/M$ between element centers is approximately 0.51 and 0.32 wavelengths respectively.

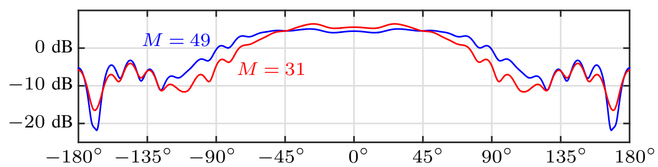


Fig. 2. Magnitude vs array-plane azimuth θ of a prototype embedded element pattern $f_0(\theta)$ constructed for each UCA design via linear interpolation from complex samples computed at 0.5° intervals using electromagnetic simulation.

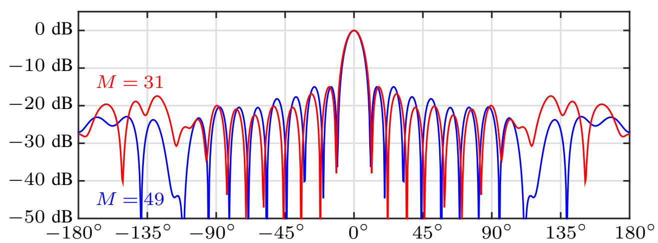


Fig. 3. UCA pattern magnitudes for designs of $M=31$ and $M=49$ elements.

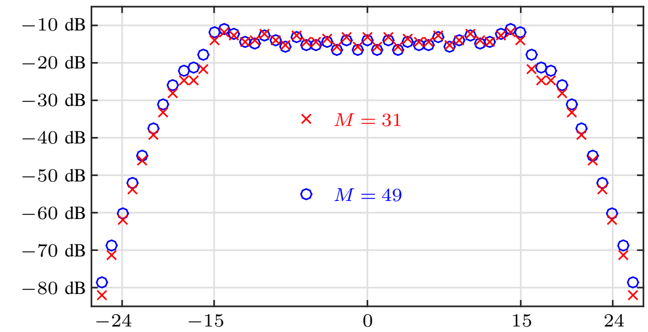


Fig. 4. Largest element-pattern Fourier coefficient magnitudes $|a_k|$ versus k.

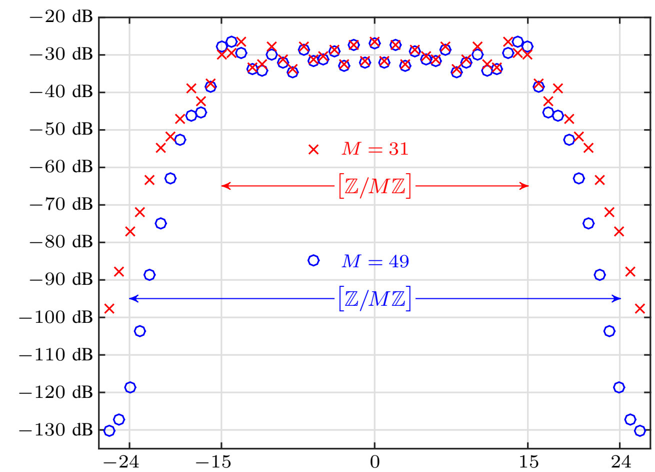


Fig. 5. Largest UCA-pattern Fourier coefficient magnitudes $|W_k a_k|$ vs k.

The first such idea, an aside here really, is that the backlobe area in its Fig. 3 pattern shows the $M=31$ design to be somewhat inadequate even before considering steering. The partial nulling as seen there generally arises from degrees of freedom that the optimization cannot quite control. We will need the Fourier-series analysis below to see that the issue here is likely too few elements or, equivalently, a elements spaced too widely. Note that issue is not one of grating lobes as it would be for a badly spaced linear or planar array.

So let us examine the several Fourier series. Figs. 4 and 5 respectively show the largest Fourier coefficient magnitudes for element-pattern series (3) and UCA-pattern series (4), for both designs. The element pattern's coefficient integrals were computed to machine accuracy, so the only approximations were in the electromagnetic simulation and in the interpolation through which we formally defined our element patterns. Our UCA patterns optimize the M values of w_m. That's equivalent, through DFT change of basis (5), to optimizing the M values of W_k for $k=0,\ldots,M-1$ or, the same by periodicity, for $k \in [\mathbb{Z}/M\mathbb{Z}]$ as constructed in Section B2 and shown in Fig. 5. We see in Fig. 5 designs that each set $[\mathbb{Z}/M\mathbb{Z}]$ happens to contain the central k, the ones nearest the origin. This is not a surprise, as those coefficients $W_k a_k$ are the largest.

The $M=31$ design has many large a_k with $k \notin [\mathbb{Z}/M\mathbb{Z}]$, so independent control of all large $W_k a_k$ is impossible. The W_k have too few degrees of freedom. Periodicity requires $W_{16} = W_{-15}$, for example, and both multiply large a_k, so two major terms in UCA pattern series (4) cannot be fixed independently. It could be worse, however, as here the a_k for $k \in [\mathbb{Z}/M\mathbb{Z}]$ all happen to be—note it is not so for the $M=49$ design—larger by several dB than any a_k for $k \notin [\mathbb{Z}/M\mathbb{Z}]$, allowing the optimization to favor W_{-15} over W_{16}. Still we conjecture that forced poor W_k choices for k just outside $[\mathbb{Z}/M\mathbb{Z}]$ led to weak nulling in the $M=31$ pattern in Fig. 3 for $|\theta| > 90°$.

Things are very different for the $M=49$ design, as $[\mathbb{Z}/M\mathbb{Z}]$ is much larger, and the W_k that through periodicity are, loosely speaking, not in direct optimization control are very small. Two large a_k never multiply the same W_k through periodicity. In fact array size $M=49$ is clearly larger than needed, as W_k for some $k \in [\mathbb{Z}/M\mathbb{Z}]$ are used to precisely fix Fig. 5 coefficients at around -120 dB. (Also, the physical array's reflection matrix appears ill conditioned with M so large.)

C. Steering the Pattern

Figs. 6 and 7 show, for the $M=31$ and $M=49$ designs respectively, the UCA magnitude patterns resulting from steering the M coefficients W_k for $k \in [\mathbb{Z}/M\mathbb{Z}]$ using the Section II procedure. When $M=31$, periodicity results in large

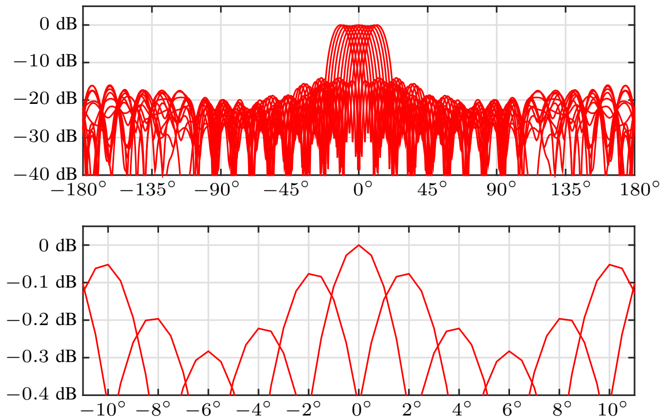


Fig. 6. The Fig. 3 $M = 31$ pattern steered in $2°$ increments. Steering Fourier coefficients indexed by $[\mathbb{Z}/M\mathbb{Z}]$ induces visible error, as $[\mathbb{Z}/M\mathbb{Z}]$ fails to include the indices of all large Fourier-series coefficients $W_k a_k$ in Fig. 5.

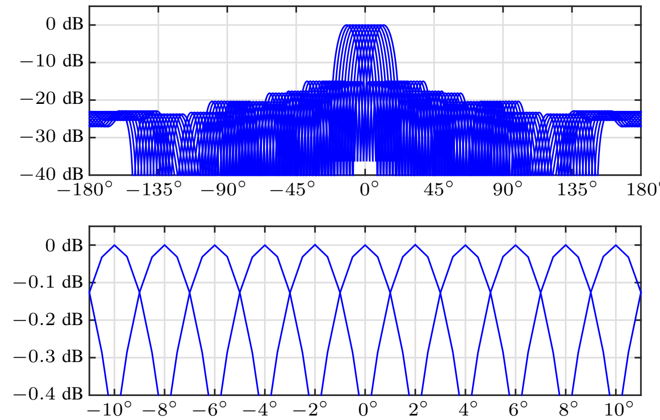


Fig. 7. The Fig. 3 $M = 49$ pattern steered in increments of $2°$. Steering Fourier coefficients indexed by $[\mathbb{Z}/M\mathbb{Z}]$ induces negligible error, as $[\mathbb{Z}/M\mathbb{Z}]$ includes the indices of all large Fourier-series coefficients $W_k a_k$ in Fig. 5.

coefficients being "missteered," causing pattern distortion. No such conflicts appear when $M = 49$. Instead steering is clean and nondistorting. The "pointy hats" on the peaks in the closeup view in Fig. 7 are not plot artifacts but result from the linear interpolation wired into our element patterns.

Steering in Figs. 6 and 7 is by $\theta_s = -12°, -10°, \ldots, 12°$ only, but Fig. 8 shows steering through the entire circle. The sidelobe distortion in the $M = 31$ left plot displays a "diagonal periodicity." To see why, note (9) implies $k' = k + \ell M$ for some integer ℓ dependent on k' and hence on k. By (10) then,

$$f'_{\theta_s + \Delta\theta}(\theta) = \sum_k W_k\, \mathrm{e}^{-jk'\theta_s}\, a_k\, \mathrm{e}^{jk(\theta - \Delta\theta)}\, \mathrm{e}^{-j\ell M \Delta\theta}\,.$$

But $\mathrm{e}^{-j\ell M\Delta\theta} = 1$ when $\Delta\theta$ is an integral multiple of $2\pi/M$, so in that case, $f'_{\theta_s+\Delta\theta}(\theta) = f'_{\theta_s}(\theta - \Delta\theta)$. In the $M = 49$ plot, the distortion amplitude is so low that this periodicity is not even visible, suggesting that $M = 49$ is higher than necessary.

D. Steering a Pattern Created by Steering a Pattern

As a final example, one perhaps more entertaining than useful, we steer the $M = 49$ design twice to create difference pattern $f_{\text{diff}}(\theta) \triangleq f'_{5°}(\theta) - f'_{-5°}(\theta)$ with the magnitude shown in Fig. 9(a). Pattern (1) is linear in the weights, so one sum using differential weights suffices. Then we create an ultra simple, classic monopulse angle measurement as the ratio of the array output with the difference weights to the array output with the original weights of the $M = 49$ design. For a single point source, the complex signal amplitude cancels to leave, for a source in the main beam, the function of position θ of the center curve in Fig. 9(b). Only the real part is shown, as the imaginary part is zero as far as the eye would see.

The rest of Fig. 9(b) shows the result of using our steering procedure yet again, this time to steer both numerator and denominator patterns together in $2°$ increments.

IV. Conclusions

A Fourier-series view of UCA patterns offers a route to sensible UCA sizing and a simple method of steering.

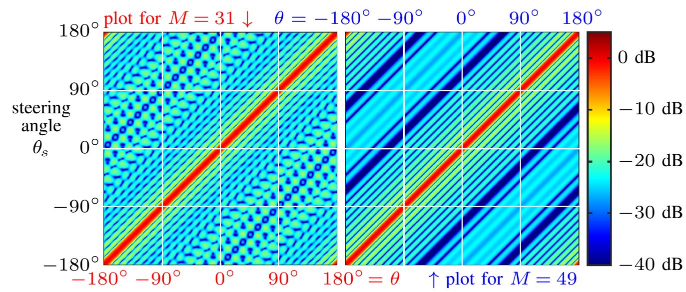
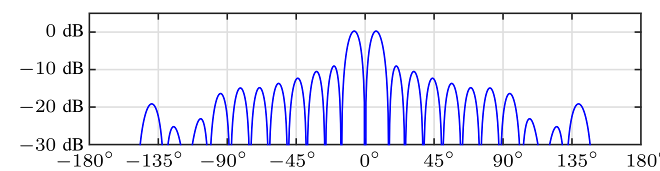
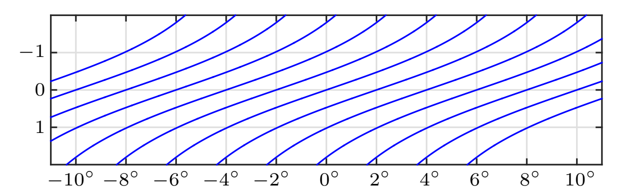


Fig. 8. UCA patterns of Fig. 3 with dB magnitude as color, plotted vs azimuth θ on the horizontal and parameterized by steering angle θ_s on the vertical.



(a) Difference-pattern magnitude.



(b) Monopulse ratio with both beams steered in $2°$ increments.

Fig. 9. Simple, steerable amplitude-monopulse bearing measurement.

References

[1] T. Taylor, "A synthesis method for circular and cylindrical antennas composed of discrete elements," *Trans. IRE Prof. Group Antennas Propag.*, vol. PGAP-3, pp. 251–261, Aug. 1952.

[2] I. N. Herstein, *Topics in Algebra*, 2nd ed. NY: Wiley, 1975.

[3] D. F. Kelley and W. L. Stutzman, "Array antenna pattern modeling methods that include mutual coupling effects," *IEEE Transactions on Antennas and Propagation*, vol. 41, no. 12, pp. 1625–1632, Dec. 1993.

[4] W. Dorsey, J. Coleman, and W. Pickles, "Uniform circular array pattern synthesis using second-order cone programming," *IET Microwaves, Antennas, and Prop.*, vol. 9, no. 8, pp. 723–727, 2015.